Competition in periodic media: pulsating fronts & segregation

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Bistable competition–diffusion system in periodic one-dimensional space

Diffusive Lotka–Volterra system

$$\begin{cases}\partial_t u_1 = \partial_{xx} u_1 + u_1 f_1(u_1, x) - k u_1 u_2\\ \partial_t u_2 = d\partial_{xx} u_2 + u_2 f_2(u_2, x) - \alpha k u_1 u_2.\end{cases}$$

- $d, k, \alpha, L > 0$,
- for any $i \in \{1,2\}$, $f_i \in \mathcal{C}^1([0,+\infty)\times\mathbb{R})$,
- f_i L-periodic w.r.t. x,
- f_i decreasing w.r.t u and $f_i(0,\bullet)$ positive.

Extinction states and bistability

Provided k is large enough, this problem has two remarkable stable periodic stationary states: the *extinction states* $(a_1, 0)$ and $(0, a_2)$.
Technical assumption: a_1 and a_2 are positive constants. Typically, $uf_i(u,x) = \mu_i(x)\, u\,(a_i - u)$.

Competition-induced segregation

The extensively studied *strong competition limit* ($k \to +\infty$) segregates the supports of u_1 and u_2, whence a free boundary problem arises [3, 4, 5].

Pulsating front solution

Definition

A solution (u_1, u_2) is a *(bistable) pulsating front* if there exists a *speed* c and a *profile* (φ_1, φ_2) such that:

- for any $(t, x) \in \mathbb{R}^2$,

$$(u_1, u_2)(t,x) = (\varphi_1, \varphi_2)(x - ct, x),$$

- φ_1 and φ_2 are resp. decreasing and increasing w.r.t. $\xi = x - ct$,
- φ_1 and φ_2 are periodic w.r.t. x,
- as $\xi \to +\infty$, uniformly w.r.t. x,

$$\begin{cases}|(\varphi_1,\varphi_2)(-\xi,x) - (a_1,0)| \to 0,\\ |(\varphi_1,\varphi_2)(\xi,x) - (0,a_2)| \to 0.\end{cases}$$

This definition follows what has been proposed in the scalar situation to generalize the concept of traveling wave [1, 2].

Existence

L, α, f_1 and f_2 are chosen such that for any value of d, there exists a family of pulsating fronts $((u_{1,k}, u_{2,k}))_{k>k^\star}$.

A consequence of our existence result

From [8], the preceding assumption is satisfied if

$$L^2 \|f_1(0,\bullet)\|_{L^\infty_{per}} \leq \pi^2.$$

Uniqueness

- The speed $c_{d,\alpha,f_1,f_2,k}$ is unique;
- The profile $(\varphi_1, \varphi_2)_{d,\alpha,f_1,f_2,k}$ is unique, up to translation w.r.t. ξ.

Importance of the sign of the speed

- **If $c > 0$, then u_1 chases u_2 and invades its habitat; conversely, if $c < 0$, u_2 chases u_1.**
- Bistable fronts are usually globally attractive for Cauchy problems with front-like initial data.

Hence, w.r.t. these Cauchy problems, the winner of the competition is entirely determined by the sign of the speed.

Question

What is the sign of the speed when the interspecific competition is large and when the two species have different diffusion rates?

Strong competition limit

Standard compactness estimates provide limiting values c_∞, $u_{1,\infty}$ and $u_{2,\infty}$.

Segregation relation

Let

$$w = \alpha u_{1,\infty} - d u_{2,\infty}.$$

Then the segregation relation is

$$\begin{pmatrix}\alpha u_{1,\infty}\\ d u_{2,\infty}\end{pmatrix} = \begin{pmatrix} w^+\\ w^-\end{pmatrix}.$$

Quasi-linear parabolic equation

Let

$$\eta : (z,x) \mapsto f_1\left(\frac{z}{\alpha}, x\right) z^+ - \frac{1}{d} f_2\left(-\frac{z}{d}, x\right) z^-,$$

$$\sigma : z \mapsto \mathbf{1}_{z>0} + \frac{1}{d}\mathbf{1}_{z<0}.$$

Then w is a non-zero sign-changing locally bounded weak solution [6] of

$$\sigma(w)\,\partial_t w - \partial_{xx} w = \eta(w, x).$$

Non-zero speeds

If $c_\infty \neq 0$, then $\alpha u_{1,\infty}$ and $-d u_{2,\infty}$ (restricted to their resp. support) are two scalar semi-pulsating fronts traveling both at the speed c_∞ and connecting resp. αa_1 to 0 and 0 to $-d a_2$.

- The free boundary condition is explicit.
- Such a solution is unique (up to translation w.r.t. ξ of its profile) and is associated with a unique speed. **Hence the whole family $(c_k)_{k>k^\star}$ converges to c_∞ as $k \to +\infty$.** w is called the *segregated pulsating front* with speed c_∞.
- The profile ϕ of w satisfies

$$-\left(\partial_{\xi\xi} + \partial_{xx} + 2\partial_{x\xi}\right)\phi - \sigma(\phi)\, c_\infty \partial_\xi \phi = \eta(\phi, x).$$

Variational formula for non-zero speeds

If $c_\infty \neq 0$, then it has the sign of

$$\int_0^L \left(\alpha^2 \int_0^{a_1} z f_1(z,x)\,dz - d\int_0^{a_2} z f_2(z,x)\,dz\right) dx.$$

Zero speeds

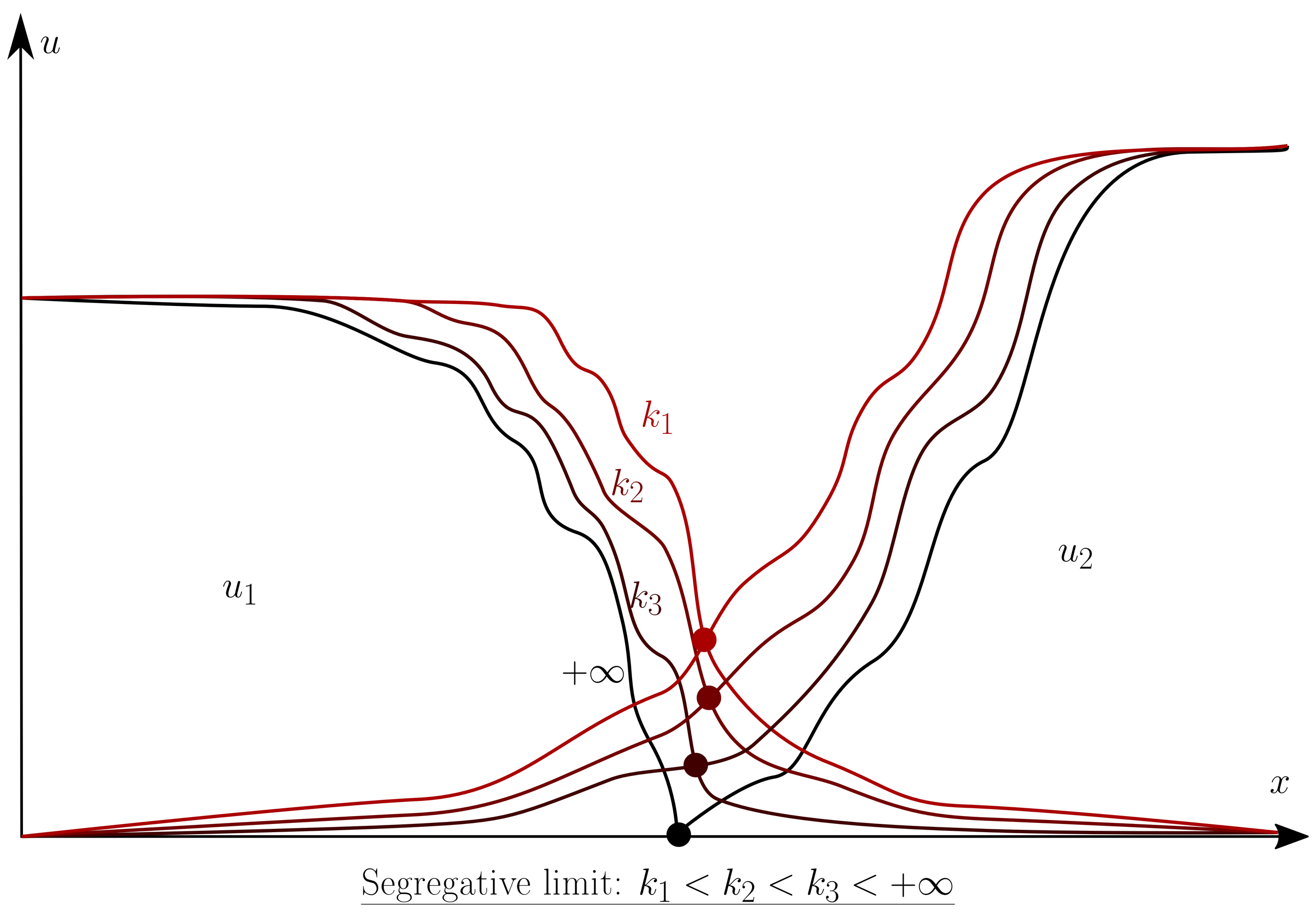
If $c_\infty = 0$, w depends only on x and is regular.

- If x_0 is a zero of w, then

$$\left(w^+\right)'(x_0) = -\left(w^-\right)'(x_0).$$

- Only w^- depends on d.

Hence provided d is small or large enough, $c_\infty \neq 0$.



Segregative limit: $k_1 < k_2 < k_3 < +\infty$

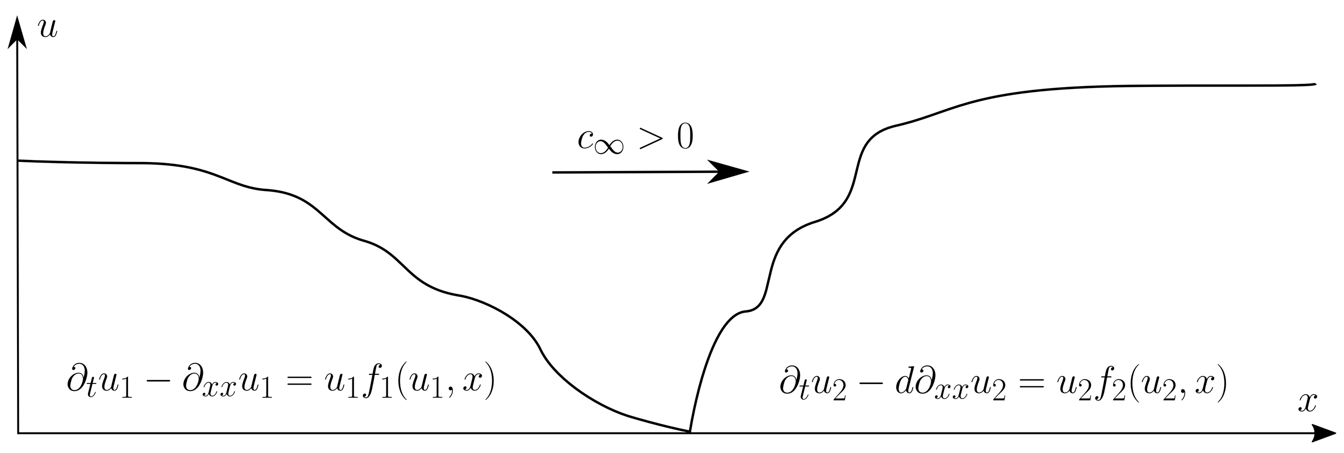
"Unity is not strength"

There exist $D^- > 0$ and $D^+ > D^-$ such that

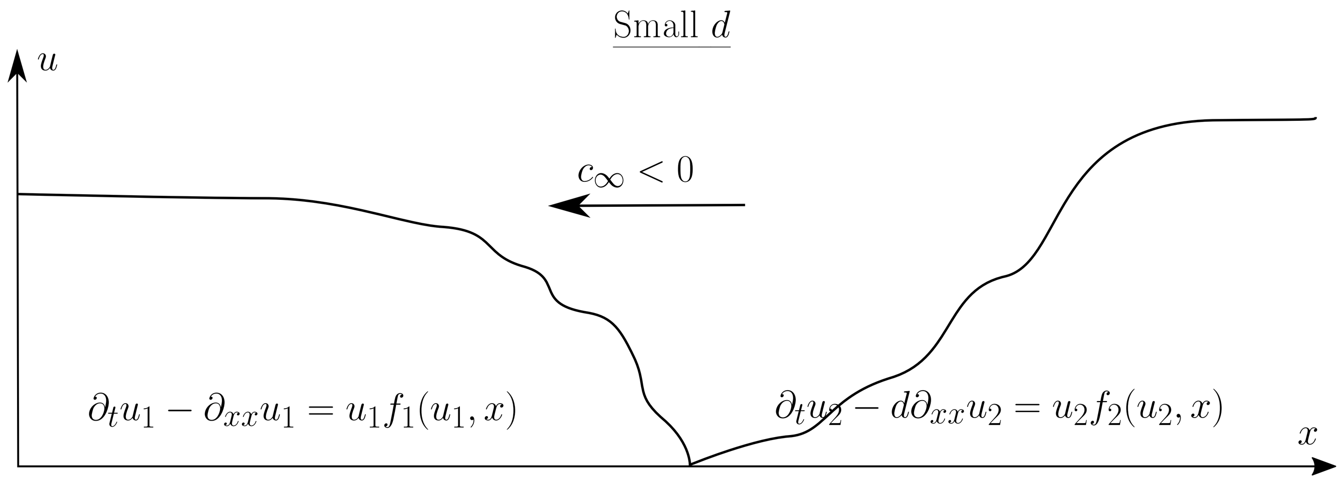
$d \in$	$(0, D^-)$	$[D^-, D^+]$	$(D^+, +\infty)$
sign (c_∞)	$+1$	?	-1

If the motilities of the two species are sufficiently contrasted, then the invader is the more motile one.

The more motile one being the more dispersed one as well, we say that, for this model, Unity is not Strength.



Small d



Large d

Conclusion

Remarks

- Extends our previous result in homogeneous media [9].
- Together with the result of Dockery, Hutson, Mischaikow and Pernarowski [7], leads to the idea that **a very motile invasive species chases a less motile resident if and only if the interspecific competition is strong.**
- Constructive proof of the existence of some scalar bistable quasi-linear fronts.
- Independent interest of the free boundary problem associated with the segregated pulsating fronts.

Open questions

- Generalization to non-constant a_1 and a_2 (variational formula for the sign of the speed non-generalizable).
- Generalization to non-positive $\min_{x\in[0,L]} f_i(0,x)$.
- Uniqueness of w when $c_\infty = 0$.
- Continuity of $\partial_t w$ and $\partial_{xx} w$ when $c_\infty \neq 0$.
- Continuity of $d \mapsto c_\infty$ in $(0,+\infty)$ (continuity in $(0, D^-) \cup (D^+, +\infty)$ established, global continuity only conjectured).

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